

# Finiteness conditions of $S$ -Cohn-Jordan Extensions

**Jerzy Matczuk\***

Institute of Mathematics, Warsaw University,  
Banacha 2, 02-097 Warsaw, Poland  
e.mail: jmatczuk@mimuw.edu.pl

## Abstract

Let a monoid  $S$  act on a ring  $R$  by injective endomorphisms and  $A(R; S)$  denote the  $S$ -Cohn-Jordan extension of  $R$ . Some results relating finiteness conditions of  $R$  and that of  $A(R; S)$  are presented. In particular necessary and sufficient conditions for  $A(R; S)$  to be left noetherian, to be left Bézout and to be left principal ideal ring are presented. This also offers a solution to Problem 10 from [6].

Throughout the paper  $R$  stands for an associative ring with unity and  $\phi$  denotes an action of a multiplicative monoid  $S$  on a ring  $R$  by injective endomorphisms. By this we mean that a homomorphism  $\phi: S \rightarrow \text{End}(R)$  is given, such that  $\phi(s)$  is an injective endomorphism of  $R$ , for any  $s \in S$ . It is assumed that all endomorphisms of  $R$  preserve unity.

We say that an over-ring  $A(R; S)$  of  $R$  is an  $S$ -Cohn-Jordan extension of  $R$  if it is a minimal over-ring of  $R$  such that the action of  $S$  on  $R$  extends to the action of  $S$  on  $A(R; S)$  by automorphisms (Cf. Definition 1.1). A classical result of Cohn (see Theorem 7.3.4 [1]) says that if the monoid  $S$  possesses a group  $S^{-1}S$  of left quotients, then  $A(R; S)$  exists, moreover it is uniquely determined up to an  $R$ -isomorphism. The above mentioned theorem of Cohn was originally formulated in much more general context of  $\Omega$ -algebras, not just rings. The construction of  $A(R; S)$  was given as a limit of a suitable directed system. The possibility of enlarging an object and replacing the action of endomorphisms by the action of automorphisms is a powerful tool, similar to a localization. This is indeed the case. One can see [3], [4], [7], [8], [9], [10] for examples of such applications in various algebraic contexts.

Jordan in [2] began systematic studies of relations between various algebraic properties of a ring  $R$  and that of  $A(R; S)$  in the case  $S = \langle \sigma \rangle$  is a monoid generated by an injective endomorphism  $\sigma$  of  $R$ . Then Matczuk in [5] started such investigations in the case  $S$  is an arbitrary monoid acting, by injective endomorphisms, on a ring  $R$ . This paper can be considered as a continuation of works carried out in [5]. One can also consult [5] and

---

\*The research was supported by Polish MNiSW grant No. N N201 268435

references inside for other motivations and examples of applications of  $S$ -Cohn-Jordan extensions.

The aim of this paper is to continue investigation of some finiteness conditions of the  $S$ -Cohn-Jordan extension  $A(R; S)$  of  $R$  in terms of properties of  $R$  and the action of  $S$ . The basic idea, which goes back to Jordan [2], is to compare left ideals  $I$  of  $A(R; S)$  with its orbits  $\{\phi(s)(I) \cap R \mid s \in S\}$  in  $R$ .

The paper is organised as follows: In a short Section 1 we present technicalities needed in the next section. Those generalize and rework Theorem 2.10 of [5] which gives a correspondence between left ideals of  $A(R; S)$  and certain admissible sets of left ideals of  $R$ . In the same time it simplifies considerations from [5] and makes the paper self contained.

In Section 2 we give necessary and sufficient conditions for  $A(R; S)$  to be noetherian (Theorem 2.4), left principal ideal ring (Theorem 2.7) and to be left Bézout ring (Proposition 2.9). Some applications and examples are presented. In particular it appears that  $A(R; S)$  is always left Bézout provided  $R$  is such. The behaviour of the noetherian property is much more complicated. Even when  $S$  is a cyclic monoid, one can find examples of rings  $R$  and  $A(R; S)$  showing that one of those rings is left noetherian but the other is not.

Theorem 2.4 gives an answer to Problem 10 posed in [6]. The characterization presented in the statement (2) of this theorem is a generalization of the one obtained by Jordan in [2], in the case when  $S = \langle \sigma \rangle$ , where  $\sigma$  is an injective endomorphism of  $R$ . However the ideas for his proof are different from ours.

## 1 Preliminaries

Henceforth  $R$  stands for an associative unital ring and  $S$  denotes a monoid which possesses a group  $S^{-1}S$  of left quotients. Recall that this is the case exactly when the monoid  $S$  is left and right cancellative and satisfies the left Ore condition. That is, for any  $s_1, s_2 \in S$ , there exist  $t_1, t_2 \in S$  such that  $t_1 s_1 = t_2 s_2$ .

Let  $\phi: S \rightarrow \text{End}(R)$  denote the action of  $S$  on  $R$  by injective endomorphisms. For any  $s \in S$ , the endomorphism  $\phi(s) \in \text{End}(R)$  will be denoted by  $\phi_s$ .

**Definition 1.1.** An over-ring  $A(R; S)$  of  $R$  is called an  $S$ -Cohn-Jordan extension ( $CJ$ -extension, for short) of  $R$  if:

1. the action of  $S$  on  $R$  extends to an action of  $S$  (also denoted by  $\phi$ ) on  $A(R; S)$  by automorphisms, i.e.  $\phi_s$  is an automorphism of  $A(R; S)$ , for any  $s \in S$ .
2. every element  $a \in A(R; S)$  is of the form  $a = \phi_s^{-1}(b)$ , for some suitable  $b \in R$  and  $s \in S$ .

As it was mentioned in the introduction, the  $CJ$ -extension  $A(R; S)$  exists and is uniquely defined up to an  $R$ -isomorphism (see also [5]).

Hereafter, as in the above definition,  $\phi_s$  will also denote the automorphism  $\phi(s)$  of  $A(R; S)$  and  $\phi_{s^{-1}}$  will stand for its inverse  $(\phi_s)^{-1}$ , where  $s \in S$ . In particular, the preimage in  $R$  of a subset  $X$  of  $R$  under the action of  $s \in S$  is equal to  $\phi_{s^{-1}}(X) \cap R$ .

**Definition 1.2.** A set  $\{X_s\}_{s \in S}$  of subsets of  $R$  is called  $S$ -admissible if, for any  $k, s \in S$ , we have  $R \cap \phi_{s^{-1}}(X_{sk}) = X_k$ . For such a set let  $\Delta(\{X_s\}_{s \in S}) = \bigcup_{s \in S} \phi_{s^{-1}}(X_s) \subseteq A(R; S)$ .

*Remark 1.3.* Let  $\{X_s\}_{s \in S}$  be an  $S$ -admissible set. Then  $\phi_s(X_k) \subseteq X_{sk}$ , for any  $k, s \in S$ . Indeed  $\phi_s(X_k) = \phi_s(R \cap \phi_{s^{-1}}(X_{sk})) \subseteq \phi_s(R) \cap X_{sk} \subseteq X_{sk}$ .

**Lemma 1.4.** Let  $X$  be a subset of  $A(R; S)$  and  $\Gamma(X) = \{X_s = \phi_s(X) \cap R\}_{s \in S}$ . Then  $\{X_s\}_{s \in S}$  is an  $S$ -admissible set of subsets of  $R$  and  $X = \bigcup_{s \in S} \phi_{s^{-1}}(X_s)$ , i.e.  $\Delta\Gamma(X) = X$ .

*Proof.* Let  $s, k \in S$ . Notice that  $R \cap \phi_{s^{-1}}(X_{sk}) = R \cap \phi_{s^{-1}}(\phi_{sk}(X) \cap R) = R \cap \phi_k(X) \cap \phi_{s^{-1}}(R) = R \cap \phi_k(X) = X_k$ , as  $R \subseteq \phi_{s^{-1}}(R)$ . This shows that  $\{X_s\}_{s \in S}$  is an  $S$ -admissible set.

The inclusion  $X \subseteq \bigcup_{s \in S} \phi_{s^{-1}}(X_s)$  is a consequence of the fact that for any  $x \in X$ , there is  $s \in S$  such that  $\phi_s(x) \in R$ . The reverse inclusion holds, since  $\phi_s$  is monic, for every  $s \in S$ .  $\square$

Notice that the set of all  $S$ -admissible sets has a natural partial ordering given by

$$\{X_s\}_{s \in S} \leq \{Y_s\}_{s \in S} \text{ if and only if } X_s \subseteq Y_s, \text{ for all } s \in S.$$

**Proposition 1.5.** There is an order-preserving one-to-one correspondence between the set  $\mathcal{L}$  of all subsets of  $A(R; S)$  ordered by inclusion and the partially ordered set  $\mathcal{R}$  of all  $S$ -admissible sets of subsets of  $R$ . The correspondence is given by maps  $\Delta$  and  $\Gamma$  defined above.

*Proof.* By Lemma 1.4, the maps  $\Delta$  and  $\Gamma$  are well-defined and satisfy  $\Delta\Gamma = \text{id}_{\mathcal{L}}$ . Clearly both maps preserve the ordering.

Let  $\{X_k\}_{k \in S}$  be an  $S$ -admissible set of subsets of  $R$ . Then

$$(1.I) \quad \{X_k\}_{k \in S} \leq \Gamma\Delta(\{X_k\}_{k \in S}) = \{Y_k\}_{k \in S}$$

where  $Y_k = R \cap \phi_k(\bigcup_{s \in S} \phi_{s^{-1}}(X_s)) = \bigcup_{s \in S} \phi_{ks^{-1}}(X_s)$ . Let  $a \in Y_k$ . Then there are  $s \in S$  and  $b \in X_s$  such that  $a = \phi_{ks^{-1}}(b)$ . Since  $S$  satisfies the left Ore condition, we can pick  $t, l \in S$  such that  $tk = ls$ . Hence  $a = \phi_{t^{-1}l}(b)$  and  $\phi_t(a) = \phi_l(b) \in \phi_l(X_s) \subseteq X_{ls} = X_{tk}$ , where the last inclusion is given by Remark 1.3. Therefore we obtain  $a \in R \cap \phi_{t^{-1}}(X_{tk}) = X_k$ , as  $\{X_s\}_{s \in S}$  is an  $S$ -admissible set. This shows that  $Y_k \subseteq X_k$ , for any  $k \in S$ . This together with (1.I) yield that  $\{X_k\}_{k \in S} = \Gamma\Delta(\{X_k\}_{k \in S})$  and complete the proof of the proposition.  $\square$

**Proposition 1.6.** Let  $A$  be an over-ring of  $R$  such that the action of  $S$  on  $R$  extends to the action of  $S$  on  $A$  by automorphisms. Then  $B = \bigcup_{s \in S} \phi_{s^{-1}}(R)$  is a  $CJ$ -extension of  $R$ .

*Proof.* Let  $a, b \in B$  and  $k, l \in S$  be such that  $\phi_k(a), \phi_l(b) \in R$ . Since  $S$  satisfies the left Ore condition, there are  $s, t, w \in S$  such that  $sk = tl = w$ . Then  $\phi_w(a) = \phi_{sk}(a), \phi_w(b) = \phi_{tl}(b) \in R$ . This implies that  $a - b, ab \in \phi_{w^{-1}}(R) \subseteq B$  and shows that  $B$  is a subring of  $A$ .

By definition of  $B$ ,  $\phi_{k^{-1}}(B) \subseteq B$  and  $B \subseteq \phi_k(B)$  follows, for any  $k \in S$ . The left Ore condition implies for any  $k, s \in S$  we can find  $l, t \in S$  such that  $ks^{-1} = t^{-1}l$ . Then  $\phi_k(\phi_{s^{-1}}(R)) = \phi_{t^{-1}}(\phi_l(R)) \subseteq \phi_{t^{-1}}(R)$ . This means that also  $\phi_k(B) \subseteq B$ , for  $k \in S$ . Now it is easy to complete the proof.  $\square$

We will say that a subset  $X$  of  $A(R; S)$  is  $S$ -invariant if  $\phi_s(X) \subseteq X$ , for all  $s \in S$ .

Direct application of Proposition 1.6 gives the following:

**Corollary 1.7.** *Let  $T$  be an  $S$ -invariant subring of  $R$ . Then  $\bigcup_{s \in S} \phi_{s^{-1}}(T) \subseteq A(R; S)$  is a  $CJ$ -extension of  $T$ .*

**Proposition 1.8.** *Let  $T$  be an  $S$ -invariant subring of  $R$  and  $B = \bigcup_{s \in S} \phi_{s^{-1}}(T) \subseteq A(R; S)$ . Let  $X$  be a subset of  $A(R; S)$  and  $\{X_s\}_{s \in S} = \Gamma(X)$ . Then:*

1.  $X$  is an additive subgroup (a subring) of  $A(R; S)$  iff for any  $s \in S$ ,  $X_s$  is an additive subgroup (a subring) of  $R$ .
2.  $X$  is a left (right)  $B$ -submodule of  $A(R; S)$  iff for any  $s \in S$ ,  $X_s$  is a left (right)  $T$ -submodule of  $R$ .

*Proof.* (1). If  $X$  is an additive subgroup (a subring) of  $A(R; S)$ , then so is  $X_s = \phi_s(X) \cap R$ , for any  $s \in S$ .

Suppose now, that  $\{X_s\}_{s \in S}$  consists of additive subgroups (subrings) of  $R$ . Let  $a, b \in X$ . Then there are  $s, t \in S$  such that  $\phi_s(a) \in X_s$  and  $\phi_t(b) \in X_t$ . By the left Ore condition of  $S$ , we can pick  $k, l \in S$  such that  $ks = lt = w$ . Then, making use of Remark 1.3, we have  $\phi_w(a) = \phi_k(\phi_s(a))$ ,  $\phi_w(b) = \phi_l(\phi_t(b)) \in X_w$ . Now it is easy to complete the proof of (1).

(2). We will prove only the left version of the statement (2). Suppose that  $X$  is a left  $B$ -submodule of  $A(R; S)$  and let  $s \in S$ . Then  $TX_s \subseteq R \cap B\phi_s(X) = R \cap \phi_s(BX) \subseteq X$ , as  $B = \phi_s(B)$ . This together with (1) show that  $X_s$  is a left  $T$ -submodule of  $R$ .

Suppose now, that  $\{X_s\}_{s \in S}$  consists of left  $T$ -submodules of  $R$ . Let  $b \in B$  and  $x \in X$ . Then there exist  $s, t \in S$  be such that  $\phi_s(b) \in T$ ,  $\phi_t(x) \in X_t$ . Since  $T$  is  $S$ -invariant, similarly as in the proof of (1), we can find  $w \in S$  such that  $\phi_w(b) \in T$  and  $\phi_w(x) \in X_w$ . Then  $\phi_w(bx) \in X_w$  and  $bx \in \phi_{w^{-1}}(X_w) \subseteq X$  follows. This together with (1) completes the proof.  $\square$

Let  $T$  be an  $S$ -invariant subring of  $R$ . We will say that an  $S$ -admissible set  $\{X_s\}_{s \in S}$  of subsets of  $R$  is an  $S$ -admissible set of left (right)  $T$ -modules if each  $X_s$  is a left (right)  $T$ -module. Propositions 1.5 and 1.8 imply the following

**Corollary 1.9.** *Let  $T$  be an  $S$ -invariant subring of  $R$  and  $B = \bigcup_{s \in S} \phi_{s^{-1}}(T) \subseteq A(R; S)$ . There is a one-to-one correspondence between the set of all left (right)  $B$ -submodules of  $A(R; S)$  and the set of all  $S$ -admissible sets of left (right)  $T$ -submodules of  $R$ .*

*Remark 1.10.* 1. If we take  $T = R$  in the above corollary, then  $B = A(R; S)$  and the corollary gives one-to-one correspondence between the set of all left, right, two-sided ideals of  $A(R; S)$  and the set of all  $S$  admissible sets of all left, right, two-sided ideals of  $R$ , respectively.

2. Let  $W, T$  be  $S$ -invariant subrings of  $R$  such that  $\bigcup_{s \in S} \phi_{s^{-1}}(W) = \bigcup_{s \in S} \phi_{s^{-1}}(T) = B \subseteq A(R; S)$  (for example assume  $S$  is commutative and take  $W = R$  and  $T = \phi_t(R)$ , for some  $t \in S$ ). Then an  $S$ -admissible set  $\{X_s\}_{s \in S}$  consists of left  $W$ -submodules iff it consists of left  $T$ -submodules as it corresponds to a  $B$ -submodule of  $A(R; S)$ . On the other hand, observe that  $T$  is a left  $T$ -module and it does not have to be a left  $W$ -module.

**Lemma 1.11.** *Let  $T$  be an  $S$ -invariant subring of  $R$ ,  $B = \bigcup_{s \in S} \phi_{s^{-1}}(T)$  its CJ-extension of  $T$  contained in  $A(R; S)$ . Then, for any subset  $X$  of  $R$  and  $k \in S$  we have  $B\phi_k(X) \cap R = \bigcup_{s \in S} \phi_{s^{-1}}(T\phi_{sk}(X)) \cap R$ .*

*Proof.* Let  $x \in B\phi_k(X) \cap R$ . Then  $x = \sum_{i=1}^n b_i \phi_k(x_i) \in R$ , where  $b_i \in B$  and  $x_i \in X$ , for  $1 \leq i \leq n$ . Let  $s \in S$  be such that  $\phi_s(b_i) \in T$ , for all  $1 \leq i \leq n$ . Then  $\phi_s(x) = \sum_{i=1}^n \phi_s(b_i) \phi_{sk}(x_i) \in T\phi_{sk}(X)$ . This shows that  $B\phi_k(X) \cap R \subseteq \bigcup_{s \in S} \phi_{s^{-1}}(T\phi_{sk}(X)) \cap R$ . The reverse inclusion is clear as, for any  $s \in S$ , we have  $\phi_{s^{-1}}(T) \subseteq B$  and  $\phi_{s^{-1}}(\phi_{sk}(X)) \subseteq \phi_k(X)$ .  $\square$

**Definition 1.12.** Let  $T, W$  be  $S$ -invariant subrings of  $R$ . For any  $(T, W)$ -subbimodule  $M$  of  $R$  and  $k \in S$  we define  $c_k^{(T, W)}(M) = \bigcup_{s \in S} \phi_{s^{-1}}(T\phi_{sk}(M)W) \cap R$ .

**Proposition 1.13.** *Let  $M$  be a  $(T, W)$ -subbimodule of  $R$ , where  $T, W$  are  $S$ -invariant subrings of  $R$  and  $B = A(T; S), C = A(W; S) \subseteq A(R; S)$ . Then  $\{c_s^{(T, W)}(M)\}_{s \in S}$  is an admissible set of  $(T, W)$ -bimodules associated to the  $(B, C)$ -subbimodule  $BMC$  of  $A(R; S)$ .*

*Proof.* Let us consider  $(B, C)$ -subbimodule  $BMC$  of  $A(R; S)$ . Since  $\phi_s(B) = B$  and  $\phi_s(C)$ , for all  $s \in S$ , we have  $\Gamma(BMC) = \{B\phi_s(M)C \cap R\}_{s \in S}$ . Now, the proof is a direct consequence of a bimodule versions of Corollary 1.9 and Lemma 1.11.  $\square$

**Definition 1.14.** Let  $M$  be a  $(T, W)$ -subbimodule of  $R$ , where  $T, W$  are  $S$ -invariant subrings of  $R$  and  $B = A(T; S), C = A(W; S) \subseteq A(R; S)$ . We say that  $M$  is  $(T, W)$ -closed if  $M = BMC \cap R$ .

The following proposition offers an internal (in  $R$ ) characterization of  $(T, W)$ -closed subbimodules of  $R$ .

**Proposition 1.15.** *For a  $(T, W)$ -subbimodule  $M$  of  $R$  the following conditions are equivalent:*

1.  $M$  is  $(T, W)$ -closed.
2.  $c_{\text{id}}^{(T, W)}(M) = M$
3.  $R \cap \phi_{s^{-1}}(T\phi_s(M)W) \subseteq M$ , for any  $s \in S$ .

*Proof.* Recall that  $c_{\text{id}}^{(T, W)}(M) = \bigcup_{s \in S} \phi_{s^{-1}}(T\phi_s(M)W) \cap R$ . The equivalence (1)  $\Leftrightarrow$  (2) is given by Proposition 1.13. The implication (2)  $\Rightarrow$  (3) is a tautology.

The statement (3) yields that  $c_{\text{id}}^{(T, W)}(M) \subseteq M$  and clearly  $M \subseteq c_{\text{id}}^{(T, W)}(M)$ . This shows that (3)  $\Rightarrow$  (2) and completes the proof of the proposition.  $\square$

Let us notice that if  $V$  is a  $(B, C)$ -subbimodule of  $A(R; S)$ , then  $V \cap R$  is a  $(T, W)$ -subbimodule of  $R$  and  $V \cap R \subseteq B(V \cap R)C \cap R \subseteq V \cap R$ , i.e.  $V \cap R$  is a  $(T, W)$ -closed subbimodule of  $R$ .

**Proposition 1.16.** *Let  $T, W$  be  $S$ -invariant subrings of  $R$ . Then:*

1. If  $\{X_s\}_{s \in S}$  is an  $S$ -admissible set of  $(T, W)$ -subbimodules of  $R$ , then  $X_s$  is a closed  $(T, W)$ -subbimodule of  $R$ , for each  $s \in S$ .
2. Let  $T_1 \subseteq T$  and  $W_1 \subseteq W$  be  $S$ -invariant subrings. Then any  $(T, W)$ -closed subbimodule  $M$  of  $R$  is closed as  $(T_1, W_1)$ -subbimodule.

*Proof.* By Corollary 1.9, there is  $(B, C)$ -subbimodule  $V$  of  $A(R; S)$  such that  $X_s = \phi_s(V) \cap R$ . This together with the observation made just before the proposition, gives (1).

The statement (2) is an easy exercise if we use the description (3) of closeness from Proposition 1.15.  $\square$

## 2 Applications

In this section we restrict our attention to left ideals, i.e. we take  $T = R$  and  $W$  is the subring of  $R$  generated by 1. In this case, for  $k \in S$ , we will write  $c_k$  instead of  $c_k^{(T, W)}$ . That is, by Proposition 1.13,  $c_k(M) = A(R; S)\phi_k(M) \cap R$ , for any left ideal  $M$  of  $R$ .

Recall (Cf. Remark 1.10)(1)) that there is one-to-one correspondence between left ideals of  $A(R; S)$  and  $S$ -admissible sets of left ideals of  $R$ . If a left ideal  $L$  of  $A(R; S)$  corresponds to the  $S$ -admissible set  $\{L_s\}_{s \in S}$ , we will say that  $L$  is associated to  $\{L_s\}_{s \in S}$  or that  $\{L_s\}_{s \in S}$  is associated to  $L$ .

**Definition 2.1.** We say that an  $S$ -admissible set  $\{L_s\}_{s \in S}$  of left ideals of  $R$  is stable if there exists  $k \in S$  such that  $c_s(L_k) = L_{sk}$ , for all  $s \in S$ .

The following proposition offers some other characterizations of stability of  $S$ -admissible sets of left ideals.

**Proposition 2.2.** Let  $\{L_s\}_{s \in S}$  be an  $S$ -admissible set of left ideals of  $R$  and  $L$  be its associated left ideal of  $A(R; S)$ . The following conditions are equivalent:

1.  $\{L_s\}_{s \in S}$  is stable.
2. There exists  $k \in S$  such that  $\phi_{sk}(L) = A(R; S)(\phi_{sk}(L) \cap R)$ , for any  $s \in S$ .
3. There exists  $k \in S$  such that  $\phi_k(L) = A(R; S)(\phi_k(L) \cap R)$ .
4. There exist  $k \in S$  and a left ideal  $W$  of  $R$  such that  $\phi_k(L) = A(R; S)W$ .

*Proof.* (1)  $\Rightarrow$  (2). Suppose  $\{L_s\}_{s \in S}$  is stable, that is we can pick  $k \in S$  such that  $c_s(L_k) = L_{sk}$ , for all  $s \in S$ . Recall that  $L_s = \phi_s(L) \cap R$ . This means that  $\{L_{sk}\}_{s \in S} = \{c_s(L_k)\}_{s \in S}$  is an  $S$ -admissible set of left ideals of  $R$  associated to  $\phi_k(L)$ . Now, Proposition 1.13 applied to  $M = L_k$ , yields that the left ideals  $\phi_k(L)$  and  $A(R; S)L_k$  of  $A(R; S)$  have the same associated  $S$ -admissible sets. Hence, by Proposition 1.5,  $\phi_k(L) = A(R; S)L_k$ . Then, for any  $s \in S$ , we have

$$\begin{aligned} A(R; S)(\phi_s(\phi_k(L)) \cap R) &\subseteq \phi_s(\phi_k(L)) = \phi_s(A(R; S)(\phi_k(L) \cap R)) \subseteq \\ &\subseteq A(R; S)(\phi_s(\phi_k(L)) \cap \phi_s(R)) \subseteq A(R; S)(\phi_s(\phi_k(L)) \cap R). \end{aligned}$$



This shows that  $\phi_{sk}(L) = A(R; S)(\phi_{sk}(L) \cap R)$ , i.e. (2) holds.

The implications (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) are tautologies.

(4)  $\Rightarrow$  (1). Let  $k \in S$  and the left ideal  $W$  of  $R$  be such that  $\phi_k(L) = A(R; S)W$ . Eventually replacing  $W$  by  $\phi_k(L) \cap R$ , we may additionally assume that  $W = \phi_k(L) \cap R = L_k$ . Therefore, by Proposition 1.13, the left ideal  $\phi_k(L)$  of  $A(R; S)$  is associated to the  $S$ -admissible set  $\{c_s(L_k)\}_{s \in S}$ . Also, by definition,  $\phi_k(L)$  is associated to  $\{\phi_s(\phi_k(L)) \cap R\}_{s \in S} = \{L_{sk}\}_{s \in S}$ . This shows that  $c_s(L_k) = L_{sk}$ , for any  $s \in S$  and completes the proof of the implication.  $\square$

**Corollary 2.3.** *Suppose that the  $S$ -admissible set  $\{L_s\}_{s \in S}$  of left ideals of  $R$  is associated to a finitely generated left ideal of  $A(R; S)$ . Then  $\{L_s\}_{s \in S}$  is stable.*

*Proof.* Let  $L = A(R; S)a_1 + \dots + A(R; S)a_n$  be a left ideal of  $A(R; S)$  associated to  $\{L_s\}_{s \in S}$  and  $k \in S$  be such that  $\phi_k(a_i) = b_i \in R$ , for  $1 \leq i \leq n$ . Then  $\phi_k(L) = A(R; S)W$ , where  $W = \sum_{i=1}^n Rb_i$ . Thus the condition (4) of Proposition 2.2 holds, i.e.  $\{L_s\}_{s \in S}$  is stable.  $\square$

Recall (Cf. Definition 1.14) that a left ideal  $X$  of  $R$  is closed if  $X = A(R; S)X \cap R$  and that  $A(R; S)X \cap R$  is always a closed left ideal of  $R$ . This implies that  $A(R; S)X \cap R$  is the smallest closed left ideal of  $R$  containing  $X$ . We will call it the closure of  $X$  and denote by  $\overline{X}$ . Proposition 1.15 offers an internal characterization of the closure of  $X$ , namely  $\overline{X} = \bigcup_{s \in S} \phi_{s^{-1}}(R\phi_s(X)) \cap R$ .

With all the above preparation we are ready to prove the following theorem.

**Theorem 2.4.** *For the  $CJ$ -extension  $A(R; S)$  of  $R$  the following conditions are equivalent:*

1.  $A(R; S)$  is left noetherian;
2. The ring  $R$  has ACC on closed left ideals and every  $S$ -admissible set of left ideals is stable;
3. Every closed left ideal of  $R$  is the closure of a finitely generated left ideal of  $R$  and every  $S$ -admissible set of left ideals is stable.

*Proof.* (1)  $\Rightarrow$  (2). Suppose  $A(R; S)$  is left noetherian. Let  $X_1 \subseteq X_2 \subseteq \dots$  be a chain of closed left ideals of  $R$ . Since  $A(R; S)$  is left noetherian, there exists  $n \geq 1$  such that  $A(R; S)X_n = A(R; S)X_{n+m}$ , for all  $m \geq 0$ . By assumption, every  $X_i$ 's is closed, so  $X_n = A(R; S)X_n \cap R = A(R; S)X_{n+m} \cap R = X_{n+m}$ , for all  $m \geq 0$ . This shows that  $R$  has ACC on closed left ideals.

Since  $A(R; S)$  is left noetherian, every  $S$ -admissible set  $\{L_s\}_{s \in S}$  of left ideals is associated to a finitely generated left ideal of  $A(R; S)$ . Hence, by Corollary 2.3,  $\{L_s\}_{s \in S}$  is stable.

(2)  $\Rightarrow$  (3). The proof is a version of a standard argument. Let  $W$  be a closed left ideal of  $R$ . Consider the set  $\mathfrak{W}$  of all closures  $\overline{I}$ , where  $I$  ranges over all finitely generated left ideals  $I$  of  $R$  contained in  $W$ . Notice that if  $\overline{I} \in \mathfrak{W}$  and  $b \in W$ , then  $\overline{I + Rb} \subseteq \overline{W} = W$ . Since  $R$  satisfies ACC on closed left ideals, we can pick a maximal element  $\overline{M}$  in  $\mathfrak{W}$  and the remark above yields  $W = \overline{M}$ .

(3)  $\Rightarrow$  (1). Let  $L$  be a left ideal of  $A(R; S)$  and  $\{L_s\}_{s \in S}$  its  $S$ -admissible set of left ideals of  $R$ . By assumption,  $\{L_s\}_{s \in S}$  is stable. Thus, by Proposition 2.2, there exist  $k \in S$  and a left ideal  $W$  of  $R$  such that  $\phi_k(L) = A(R; S)W$ . Replacing  $W$  by  $\overline{W}$  we may additionally suppose that  $W$  is closed. Then, by assumption, there exist  $b_1, \dots, b_n \in R$  such that  $W = \overline{Rb_1 + \dots + Rb_n}$ . Notice that  $A(R; S)b_1 + \dots + A(R; S)b_n \subseteq \phi_k(L) = A(R; S)(R \cap (A(R; S)b_1 + \dots + A(R; S)b_n)) \subseteq A(R; S)b_1 + \dots + A(R; S)b_n$ . This shows that  $\phi_k(L)$  is a finitely generated left ideal of  $A(R; S)$ . Since  $\phi_k$  is an automorphism of  $A(R; S)$ ,  $L$  is also finitely generated.  $\square$

The above theorem gives immediately:

**Corollary 2.5.** *Suppose that  $R$  left noetherian. Then  $A(R; S)$  is left noetherian iff every  $S$ -admissible set of left ideals of  $R$  is stable.*

The equivalence (1)  $\Leftrightarrow$  (2) in Theorem 2.4 is a generalization of Theorem 5.6 [2] from the case when the monoid  $S$  is cyclic to the case when  $S$  is a cancellative monoid satisfying the left Ore condition. The idea of the presented proof is completely different from the one used in [2].

It is known that there exist rings  $R$  such that only one of  $R$  and  $A(R; S)$  is left noetherian. The following example, which offers such rings, is a variation of examples from [2].

**Example 2.6.** 1. Let  $\sigma$  be the endomorphism of the polynomial ring  $\mathbb{Z}[x]$  given by  $\sigma(x) = 2x$ . One can check that  $A(\mathbb{Z}[x]; \langle \sigma \rangle) = \mathbb{Z} + \mathbb{Z}[\frac{1}{2}][x]x$  is not noetherian.  
2. Let  $A$  denote the field of rational functions in the set  $\{x_i\}_{i \in \mathbb{Z}}$  of indeterminates over a field  $F$  and  $\sigma$  be the  $F$ -endomorphism of  $R = F(x_i \mid i \leq 0)[x_i \mid i > 0]$  given by  $\sigma(x_i) = x_{i-1}$ , for  $i \in \mathbb{Z}$ . Then  $R$  is not noetherian and  $A = A(R; \langle \sigma \rangle)$  is a field.

The following theorem offers necessary and sufficient conditions for  $A(R; S)$  to be left principal ideal ring.

**Theorem 2.7.** *For the  $CJ$ -extension  $A(R; S)$  of  $R$  the following conditions are equivalent:*

1. *Every left ideal of  $A(R; S)$  is principal;*
2. *Every  $S$ -admissible set  $\{L_s\}_{s \in S}$  of left ideals of  $R$  satisfies the following conditions:*
  - (a)  *$\{L_s\}_{s \in S}$  is stable,*
  - (b) *There exist  $t \in S$  and  $b \in R$  such that  $L_t = \overline{Rb}$ .*

*Proof.* (1)  $\Rightarrow$  (2). Let  $\{L_s\}_{s \in S}$  be an  $S$ -admissible set of left ideals of  $R$  and  $L$  be its associated left ideal of  $A(R; S)$ . Since every left ideal of  $A(R; S)$  is principal, Corollary 2.3 implies that the property (a) holds.

Let  $a \in A(R; S)$  and  $t \in S$  be such that  $L = A(R; S)a$  and  $b = \phi_t(a) \in R$ . Then  $L_t = \phi_t(L) \cap R = A(R; S)b \cap R = \overline{Rb}$ , i.e. the property (b) is satisfied.

(2)  $\Rightarrow$  (1). Let  $L$  be a left ideal of  $A(R; S)$  and  $\{L_s\}_{s \in S}$  be its associated  $S$ -admissible set of left ideals of  $R$ . By assumption,  $\{L_s\}_{s \in S}$  is stable. Thus, applying Proposition 2.2(2), we can pick  $k \in S$  such that  $\phi_{sk}(L) = A(R; S)L_{sk}$ , for any  $s \in S$ . Observe that



$\{L_{sk}\}_{s \in S} = \{\phi_s(\phi_k(L)) \cap R\}_{s \in S}$  is an  $S$ -admissible set of left ideals associated to  $\phi_k(L)$ . Therefore we can apply (2)(b) to  $\{L_{sk}\}_{s \in S}$  and pick  $l \in S$  and  $b \in R$  such that  $L_{lk} = \overline{Rb}$ . Let us set  $t = lk$ . Using the above we have  $\phi_t(L) = A(R; S)L_t$  and  $A(R; S)b \subseteq A(R; S)L_t = A(R; S)\overline{Rb} \subseteq A(R; S)b$ . This shows that  $\phi_t(L) = A(R; S)b$  and proves that the left ideal  $L = A(R; S)\phi_{s^{-1}}(b)$  is principal.  $\square$

*Remark 2.8.* 1. It is not difficult to prove that the condition (2)(b) of the above theorem is equivalent to the condition that every closed left ideal  $X$  of  $R$  is of the form  $X = \phi_{t^{-1}}(\overline{Rb}) \cap R$ , for suitable  $t \in S$  and  $b \in R$ .

2. Let us remark that the condition (2)(b) always holds, provided every closed left ideal is principal.

Recall that a ring  $R$  is left Bézout if every finitely generated left ideal of  $R$  is principal.

**Proposition 2.9.** *For the  $CJ$ -extension  $A(R; S)$  of  $R$  the following conditions are equivalent:*

1.  $A(R; S)$  is a left Bézout ring;
2. for every  $S$ -admissible set  $\{L_s\}_{s \in S}$  associated to a finitely generated left ideal  $L$  of  $A(R; S)$ , there exist  $t \in S$  and  $b \in R$  such that  $L_t = \overline{Rb}$ .

*Proof.* Let  $L$  be a finitely generated left ideal of  $A(R; S)$  and  $\{L_s\}_{s \in S}$  its associated  $S$ -admissible set.

If  $A(R; S)$  is left Bézout, then  $L$  is principal. Thus there is  $t \in S$  and  $b \in R$  such that  $\phi_t(L) = A(R; S)b$  and  $L_t = \phi_t(L) \cap R = \overline{Rb}$ . This shows that (1) implies (2).

Suppose (2) holds. Then, by Corollary 2.3,  $\{L_s\}_{s \in S}$  is stable.

Now one can complete the proof as in the proof of implication (3)  $\Rightarrow$  (1) of Theorem 2.7.  $\square$

Notice that the characterization obtained in the above proposition is not nice in the sense that the statement (2) is not expressed in terms of properties of  $R$  but  $A(R; S)$  is involved. Anyway it has the following direct application:

**Corollary 2.10.** *Suppose that one of the following conditions is satisfied:*

1. Every closed left ideal of  $R$  is principal.
2.  $R$  is left Bézout.

*Then  $A(R; S)$  is a left Bézout ring.*

*Proof.* Proposition 2.9 and Remark 2.8(2) give the thesis when (1) holds.

Suppose (2) holds. Let  $L = A(R; S)a_1 + \dots + A(R; S)a_n$  and  $t \in S$  be such that  $b_i = \phi_t(a_i) \in R$ ,  $1 \leq i \leq n$ . By assumption, there exists  $b \in R$  such that  $Rb_1 + \dots + Rb_n = Rb$ . Then  $L_t = \phi_t(L) \cap R = A(R; S)b \cap R = \overline{Rb}$  and the thesis is a consequence of Proposition 2.9.  $\square$

The following example offers a principal ideal domain  $R$  such that  $A(R; S)$  is not noetherian. Of course, by Corollary 2.10,  $A(R; S)$  is left Bézout.

**Example 2.11.** Let  $A = K[x^{\frac{1}{2^n}} \mid n \in \mathbb{N}]$ , where  $K$  is a field, and  $\sigma$  be a  $K$ -linear automorphism of  $A$  defined by  $\sigma(x) = x^2$ . Then the restriction of  $\sigma$  to  $R = K[x]$  is an endomorphism of  $R$  and it is easy to check that  $A$  is a  $CJ$ -extension of  $R$  with respect to the action of  $\sigma$ . Notice that  $A$  is not noetherian but it is Bézout, by the above corollary.

In view of Theorem 2.7 and Proposition 2.9 it seems interesting to know when all principal left ideals of  $R$  are closed. We will concentrate on this problem till the end of the paper. It is known (Cf. Lemma 1.16 and Theorem 2.24 of [5]) that if  $R$  is a semiprime left Goldie ring, then:

- (i) every regular element  $c$  of  $R$  is regular in  $A(R; S)$ ;
- (ii)  $A(R; S)$  is a semiprime left Goldie ring and  $Q(A(R; S)) = A(Q(R); S)$ , where  $Q(B)$  denotes the classical left quotient ring of a left Goldie ring  $B$ .

Therefore both  $Q(R)$  and  $A(R; S)$  are over-rings of  $R$  included in  $A(Q(R); S)$ . Keeping the above notation we have:

**Proposition 2.12.** *For a semiprime left Goldie ring  $R$ , the following conditions are equivalent:*

1.  $Q(R) \cap A(R; S) = R$ ;
2.  $Rc = \overline{Rc}$  and  $cR = \overline{cR}$ , for every regular element  $c \in R$ ;
3.  $cR = \overline{cR}$ , for every regular element  $c \in R$ ;
4. If  $ca \in R$ , then  $a \in R$ , provided  $a \in A(R; S)$  and  $c \in R$  is regular.

*Proof.* Let  $c \in R$  be a regular element.

(1)  $\Rightarrow$  (2) Let  $a \in A(R; S)$  be such that  $ac = r \in R$ . Then  $a = rc^{-1} \in Q(R) \cap A(R; S) = R$ . This shows that  $A(R; S)c \cap R \subseteq Rc$  and implies that  $Rc = \overline{Rc}$ . A similar argument works for showing that  $cR = \overline{cR}$ .

The implication (2)  $\Rightarrow$  (3) is a tautology.

(3)  $\Rightarrow$  (4) Suppose  $ca \in R$ , where  $a \in A(R; S)$ . By (3) we have  $cR = \overline{cR} = cA(R; S) \cap R$ . thus there exists  $r \in R$  such that  $ca = cr$  and  $a = r \in R$  follows, as  $c$  is regular in  $A(R; S)$ .

(4)  $\Rightarrow$  (1) Let  $r \in R$  be such that  $c^{-1}r = a \in Q(R) \cap A(R; S)$ . The condition (4) gives  $a \in R$  and shows that  $Q(R) \cap A(R; S) = R$ .  $\square$

The statement (2) in the above proposition is left-right symmetric thus, additionally assuming that the semiprime ring  $R$  is also right Goldie, we can add to the proposition left versions of statements (3) and (4). However, as the following example shows, we can not do this when  $R$  is not right Goldie.

**Example 2.13.** Let  $D$  denote the field of fractions of the ring  $K[x^{\frac{1}{2^n}} \mid n \in \mathbb{N}]$  from Example 2.11 and  $\sigma$  be a  $K$ -linear automorphism of  $D$  defined by  $\sigma(x) = x^2$ . Let us consider the skew polynomial ring of endomorphism type (with coefficients written on the left)  $A = D[t; \sigma]$ . Then  $\sigma$  can be extended to an automorphism of  $A$  by setting  $\sigma(t) = t$ . Let  $R = K(x)[t; \sigma] \subseteq A$ . Then the restriction of  $\sigma$  to  $R$  is an endomorphism of  $R$  and

for any  $w \in A$ , there exists  $n \geq 1$  such that  $\sigma^n(w) \in R$ . This means that  $A = A(R; \langle \sigma \rangle)$ , where  $\langle \sigma \rangle$  denotes the monoid generated by  $\sigma$ .

It is well known that  $R$  is a left Ore domain which is not right Ore. Observe that  $t\sqrt{x} = xt \in R$ , but  $\sqrt{x} \notin R$ . Thus, by Proposition 2.12,  $Q(R) \cap A \neq R$ . In fact, the left localization of  $R$  with respect the left Ore set consisting of all powers of  $t$  is equal to  $D[t, t^{-1}, \sigma]$ . Thus  $A \subseteq D[t, t^{-1}, \sigma] \subseteq Q(R)$ .

We claim that  $R$  satisfies the left version of statement (4) from Proposition 2.12. Let  $0 \neq c \in R$  and  $a \in A$  be such that  $ac \in R$ . If  $a \notin R$ , then we can choose such  $a = \sum_{i=0}^n a_i t^i$  of minimal possible degree, say  $n$ . Then, by the choice of  $n$ ,  $a_n \notin K(x)$ . Then also  $a_n \sigma^n(c_n) \notin K(x)$ , where  $c_n$  denotes the leading coefficient of  $c \in R = K(x)[t; \sigma]$  and then  $ac \notin R$ , which is impossible. Thus  $a$  has to belong to  $R$ .

Observe that the ring from Example 2.11 satisfies the assumption of the following proposition.

**Proposition 2.14.** *Suppose  $R$  is a left Ore domain such that  $Q(R) \cap A(R; S) = R$ . For a left ideal  $L$  of  $A$  the following conditions are equivalent:*

1.  $L$  is principal;
2.  $\exists_{s \in S} \exists_{a \in R}$  such that, for all  $t \in S$ ,  $\phi_{ts}(L) \cap R = L_{ts} = R\phi_t(a)$ .

*Proof.* (1)  $\Rightarrow$  (2). Suppose  $L = Ab$  and let  $s \in S$  be such that  $\phi_s(a) \in R$ . Then  $\phi_s(L) = A\phi_s(b)$ . Set  $a = \phi_s(b)$ . Now the implication is a direct consequence of Proposition 2.12.

(2)  $\Rightarrow$  (1). Let  $s \in S$  and  $a \in R$  be as in (2). Then  $\phi_s(L)_t = L_{ts}R\phi_t(a)$ . This means that  $\phi_s(L) = \bigcup_{t \in S} \phi_t^{-1}(R)a = A(R; S)a$ . Thus  $L = A(R; S)\phi_s^{-1}(a)$ .  $\square$

## References

- [1] P.M. Cohn, *Universal Algebra*, Harper and Row, (1965).
- [2] D.A. Jordan, *Bijjective Extensions of Injective Rings Endomorphisms*, J. London Math. Soc. 25 (1982), 435–448.
- [3] N.S. Larsen, *Crossed Products by Semigroups of Endomorphisms and Groups of Partial Automorphisms*, Bull. Canad. Math. 46(1) (2003), 98–112.
- [4] A. Leroy, J. Matczuk, *Goldie Conditions for Ore Extensions over Semiprime Rings*, Algebras and Representation Theory 8 (2005), 679–688.
- [5] J. Matczuk, *S-Cohn-Jordan Extensions*, Comm. Algebra 35, (2007), 725–746.
- [6] J. Matczuk, *On S-Cohn-Jordan extensions*, Proc. 39-th Symposium on Ring Theory and Representation Theory, Hiroshima ed. M. Kutami, Yamaguchi, (2007), 30–35.

- [7] V.A. Mushrub, *On The Goldie Dimension of Ore Extensions With Several Variables*. Fundam. Prikl. Mat. 7(4) (2001), 1107–1121.
- [8] D. Pask, T. Yeend, *Voltage Graphs*, Voltage Graphs. Proc. Centre Math. and Appl., Austral. Nat. Univ, (1999).
- [9] D. Pask , I. Raeburn , T. Yeend, *Actions of Semigroups on Directed Graphs and their  $C^*$ -algebras*, Pure Appl. Alg. 159 (2001), 297–313
- [10] G. Picavet, *Localization with Respect to Endomorphisms*, Semigroup Forum 67 (2003), 76–96.